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A GENERAL THEORY FOR LAMINATED PLATES WITH DELAMINATIONS

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Abstract—An approximate analytical model for the behavior of a laminated composite plate in the presence of delaminations and other local effects is presented. The model is based on a generalized displacement formulation implemented at the layer level. The governing equations for a layer are obtained using the principle of virtual work. These governing equations for a layer are used in conjunction with the explicit satisfaction of both the interfacial traction continuity and the interfacial displacement jump conditions between layers to develop the governing equations for a laminated composite plate, including delaminations. The fundamental unknowns in the theory are the displacements in the layers and the interfacial tractions. The theory is sufficiently general that any constitutive model for the interfacial fracture (i.e. delamination) as well as for the layer behavior can be incorporated in a consistent fashion into the theory. The interfacial displacement jumps are expressed in an internally consistent fashion in terms of the fundamental unknown interfacial tractions. The current theory imposes no restrictions on the size, location, distribution, or direction of growth of the delaminations. Therefore, the theory can predict the initiation and growth of delaminations at any location as well as interactive effects between delaminations at different locations within the laminate.

Pagano's exact solution for the cylindrical bending of a laminated plate has been modified to include the effects of delamination. An interface model, which expressed the displacement jump as a linear function of the surface tractions, was implemented into this modification of the exact solution. This extension was used to validate the approximate plate theory. The correlation between the approximate approach and the exact solution is seen to be excellent. The approximate plate theory is seen to give very accurate predictions for the interfacial tractions in a direct and consistent fashion, i.e. without the need to use integration of the pointwise equilibrium equations. This allows the interfacial displacement jumps in the presence of delaminations to be modeled accurately. It is seen that these displacement jumps have a significant effect on both the macroscopic and microscopic behavior of a laminated plate. © 1997 Published by Elsevier Science Ltd.

1. INTRODUCTION

The use of laminated composite structures has many potential applications in a variety of engineering fields. In particular, laminated structures where the individual lamina are composed of heterogeneous materials, such as continuous fiber composites, can be particularly useful. These types of composite systems can exhibit many favorable characteristics such as high specific modulus and strength, resistance to fatigue and damage, low specific density, and directional properties. In particular, the directional properties of fibrous composites can be used to tailor various aspects of the structural response to obtain a desired behavior. However, before these potential applications can be realized it is necessary to develop analytical tools capable of accurately predicting the behavior of these structures. In general, the complexity of such structures and the applied loading makes solutions based on an exact analysis of the appropriate continuum equations impossible. This implies the need for accurate approximate models. One possibility is to use a three-dimensional finite element (FE) approach to obtain the response. In situations where a given sublaminate sequence is repeated frequently, homogenization techniques can make such an analysis feasible. However, if it is necessary to resolve the lamina microstructure, which is the case in many applications, the FE approach can be prohibitively expensive. In these situations, the relative dimensions of the lamina thickness as compared to the structural dimensions impose limitations on the element size due to aspect ratio considerations. These limitations can require a fine finite element mesh to analyze a structure. Such fine meshes can make the problem prohibitively expensive or difficult to run. In situations where homogenization

is not applicable, analyses based on plate and shell formulations are particularly attractive. These approaches model the behavior through the thickness of the structure explicitly and therefore are not restricted by aspect ratio limitations imposed by the thicknesses of the individual lamina.

In developing such an approximate plate or shell model, it is necessary to recognize that actual laminated structures exhibit a wide variety of responses. These responses can be loosely separated into two length scales. The global responses, such as the structural deformations, can be associated with a macroscopic length scale, which is of the order of the laminate thickness or some other characteristic structural dimension. The second class of responses are considered to be local effects. Local responses, which include delamination, intraply cracking, plasticity, fibre/matrix debonding, and constituent damage, occur at a microscopic length scale and can be associated with the lamina thickness or the fiber dimensions. It must be appreciated that both the global and local responses are mutually influencing. In general, the local or microscopic effects are history-dependant phenomena, which evolve in a highly nonlinear fashion. This implies the need for an accurate determination of the local fields to correctly predict the evolution of the local effects and, thus, of their nonlinear influence on the macroscopic structural response. Additionally, it must be recognized that both the local and the global effects can result in significant variation in the stress field through the thickness of a layer.

Much of the early work on laminated structures has been associated with predicting the macroscopic behavior of the structure. Typically, it was assumed that each displacement component throughout the entire thickness of the plate could be approximated by a single expansion in the thickness coordinate. The resulting theories model the subsequent behavior of the laminated structure as an equivalent anisotropic plate or shell with "smeared" material properties. This approach has met with various degrees of success in predicting the macroscopic response of laminated structures in idealized situations (Aboudi and Benveniste, 1984; Whitney, 1987; Lo *et al.*, 1977a,b; Khdeir and Reddy, 1991). However, the use of such theories can result in significant errors in the predicted response of thick structures or for structures which exhibit local phenomena. In particular, significant variations of the local fields from the assumed smeared distributions may occur in reality (Pagano, 1969, 1970a,b). When local effects are present, these variations lead to inaccurate predictions of their nonlinear evolution. This can lead to incorrect predictions for the macroscopic response of the structure. Therefore, to accurately predict the general behavior of a laminated composite in the presence of local effects any model must consider behavior of the structure at a more refined level. One possibility is to develop a model at the lamina level (Pagano, 1978; Reddy, 1987; Liu *et al.*, 1994; Chang *et al.*, 1995; Whitney, 1995). The advantage of developing a model for laminated structures at this level is that significantly less smearing of the local fields is used, which provides more accurate predications for the evolution of the local phenomena. Additionally, an approximate analysis applied at the lamina level can potentially satisfy both the interfacial traction continuity and interfacial displacement jump conditions. This ability is necessary to predict delamination.

Refined theories for the approximate analysis of laminated structures have been traditionally developed using variational methods based on through the thickness approximations for the stresses, the displacements, or both the stress and displacements within either a layer or the laminate (Lo *et al.*, 1977a,b; Pagano, 1978; Reddy, 1989; Change *et al.*, 1995; Whitney, 1995). The current theory is based on an assumed displacement field.

In developing a displacement based theory for modeling the behavior of delaminated structures several considerations arise. Typically, delaminations develop in a structure under the application of the service loads. This implies that, in general, the delamination sizes and distributions are not known *a priori*. An additional consideration is that the presence of delaminations in one part of the structure may cause unloading in adjacent regions. Therefore, other regions of the structure must support increased stresses. These increased loading states can result in the initiation and growth of delaminations in these regions. Furthermore, this shifting of the local stress states implies that the continued propagation of a given delamination in a given direction can not be assumed in any analysis *a priori*. The above issues imply that to effectively model the behavior of a laminated

structure in the presence of delaminations both the initiation and growth of the delamination must be accurately predicted. A scanning electron micrograph for the tensile region of a four-point bend specimen of a multidirectional laminate is provided in Fig. 1. Cracks which have propagated along the interface between the lamina and the resin-rich region may be observed. This cracking results in a delamination mode of failure for the composite structure.

The constitutive behavior for the fracture of interfaces, such as the interface between lamina within a composite structure, can be modeled by the following general form:

$$\boldsymbol{\Delta} = f(\boldsymbol{\Delta}, \mathbf{t}) \tag{1}$$

where the vector $\boldsymbol{\Delta}$ is the jump in the displacement field across the interface, $\mathbf{t} = \mathbf{n} \cdot \sigma$ is the interfacial traction vector, and $\mathbf{n}$ is the unit normal vector to the interface. In general, the constitutive relation for the jumps in the displacements, $f(\boldsymbol{\Delta}, \mathbf{t})$, is a nonlinear function of $\boldsymbol{\Delta}$ and $\mathbf{t}$ (Needleman, 1987, 1990; McGee and Herakovich, 1992; Corigliano, 1993). The case of perfect bonding is given by $\boldsymbol{\Delta} = 0$. It is noted that this case is the one treated by most plate theories. A completely delaminated situation (i.e. no interaction effects exist between adjacent lamina) occurs when $\boldsymbol{\Delta} \to \infty$. In general, a delamination constitutive relation, eqn (1), exhibits a transitionary behavior between these two extremes. An accurate assessment of the interfacial traction condition is necessary to correctly predict delamination. Typically, displacement based theories do not give a consistent, accurate prediction of the interfacial tractions directly within the context of the theory. The traditional method for obtaining these tractions in static, perfectly bonded laminate problems is to solve for the displacement field within the context of the theory and, subsequently, to integrate the corresponding pointwise equilibrium relations to obtain the interfacial tractions. However, this approach is not always feasible, examples being dynamic loading conditions or behavior in the presence of delaminations. In the case of delamination, the interfacial tractions and the displacement jumps are nonlinearly interdependent. Therefore an internally consistent approach to obtaining these tractions is necessary. One possibility is to incorporate these effects directly into the formulation of the model for the laminated structure.

In the current work, a formulation for a generalized theory of laminated plates in the presence of delaminations is presented. The governing equations for the response of a single layer are developed. These equations are subsequently coupled through the explicit imposition of both interfacial traction continuity and the jump conditions on the interfacial displacements to develop the governing equations for a laminate. The basic variables of the resulting theory are the layer displacements and the interfacial traction effects between layers. By treating the interfacial tractions as fundamental unknowns an internally consistent evaluation of these effects is directly obtained from the formulation, i.e. without the need to resort to the evaluation of the pointwise equilibrium equations. The formulation is not restricted to the analysis of preexisting delaminations and can predict the initiation and growth of delaminations. Additionally, the model is formulated in a sufficiently general fashion that any interfacial fracture model can be incorporated in an internally consistent fashion. The previously mentioned local effects can be modeled with any desired accuracy because no restrictions are imposed on the layer constitutive relations or the order of the approximation of the fields through the thickness of the layers. Finally, the resulting theory is completely general with respect to the thickness of the laminated structure.

The proposed formulation is validated using exact solutions for the response of both perfectly bonded and delaminated composite plates subjected to cylindrical bending. The solution for the response of a delaminated plate subjected to cylindrical bending was developed for this work and is a modification of Pagano's exact solution (1969). A linear function of the surface tractions, eqn (1), is used for the interface model in this investigation. It will be shown that the approximate formulation provides excellent agreement with the exact solution. In particular, the incorporation of the interfacial traction effects as a fundamental unknown allows the displacement jumps at interfaces to be predicated accurately.

2. GENERAL FORMULATION

The following notational conventions are used throughout the formulation. Superscripts will denote the number (or order) of the approximation function. Subscripts denote tensorial quantities. Greek subscripts have a range of x and y. Roman subscripts have a range of x, y, and z. Repeated Roman superscripts and subscripts imply summation over the appropriate range. A comma denotes differentiation with respect to the spatial coordinates. A dot denotes differentiation with respect to time.

Consider a computational layer within the composite structure, Fig. 2. It is assumed that the displacement field within this layer is approximated by

$$u_i(x, y, z, t) = V_i^j(x, y, t)\phi^j(z) \tag{2}$$

where $j = 1, 2, \ldots, N$. N is the order of the polynomial expansion. This is the same displacement approximation used by Reddy (1987) to model perfectly bonded laminates. The functions $\phi^j(z)$ are specified functions of the transverse coordinate z and the $V_i^j(x, y, t)$ are the associated displacement coefficients. The assumption given by eqn (2) admits any order of approximation because no restriction is placed on the order or functional form of the $\phi^j(z)$. The corresponding strain field within the layer due to the above approximation is given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

where the displacement field (u_i) is provided by eqn (2). The formulation of the governing equations for the layer are obtained from the application of the principle of virtual work (Washizu, 1968; Reddy, 1984)

$$\int_0^{t_1}\left(\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,V - \int_S t_i^*\,\delta u_i\,\mathrm{d}S - \int_V f_i\,\delta u_i\,\mathrm{d}V - \int_V \rho\ddot{u}_i\,\delta u_i\,\mathrm{d}V\right)\mathrm{d}t = 0 \tag{4}$$

where σ_{ij} denote the Cauchy stresses, ε_{ij} denote the strains, t_i^* denote the surface tractions, f_i are the body forces, and ρ is the density. In the above equation, the appropriate volume to be used is that of the layer given by $V = t_k \cdot \Omega$, Fig. 2. The thickness of the kth layer is given by $t_k = z^{k+1} - z^k$, where z^{k+1} and z^k are the coordinates of the top and bottom of the layer, respectively. Ω is an arbitrary area corresponding to the midplane of the layer. S is the outer surface area of the volume V.

Substituting eqns (2) and (3) into eqn (4) and integrating through the thickness (z) to obtain an equivalent two-dimensional theory gives

$$\int_0^{t_1}\left(\oint_{\partial\Omega}(T_i^j - N_{i\alpha}^j n_\alpha)\,\delta V_i^j\,\mathrm{d}s + \int_\Omega (\tau_i^j + N_{i\alpha,\alpha}^j - R_i^j + F_i^j - I^{mj}\ddot{V}_i^m)\,\delta V_i^j\,\mathrm{d}\Omega\right)\mathrm{d}t = 0$$

where $\partial\Omega$ denotes the boundary of Ω and $\mathrm{d}s$ is the associated differential arc length. Consequently, the appropriate equations of motion for the layer are

$$\tau_i^j + N_{i\alpha,\alpha}^j - R_i^j + F_i^j = I^{mj}\ddot{V}_i^m \tag{5}$$

where $m, j = 1, 2, \ldots, N$. The corresponding inplane boundary conditions are

$$\begin{aligned} V_i^j &= 0 \quad \text{on} \quad \partial\Omega_1 \\ T_i^j &= N_{i\alpha}^j n_\alpha \quad \text{on} \quad \partial\Omega_2 \end{aligned} \tag{6}$$

where $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$. The first set of conditions in eqn (6) represent the essential boundary

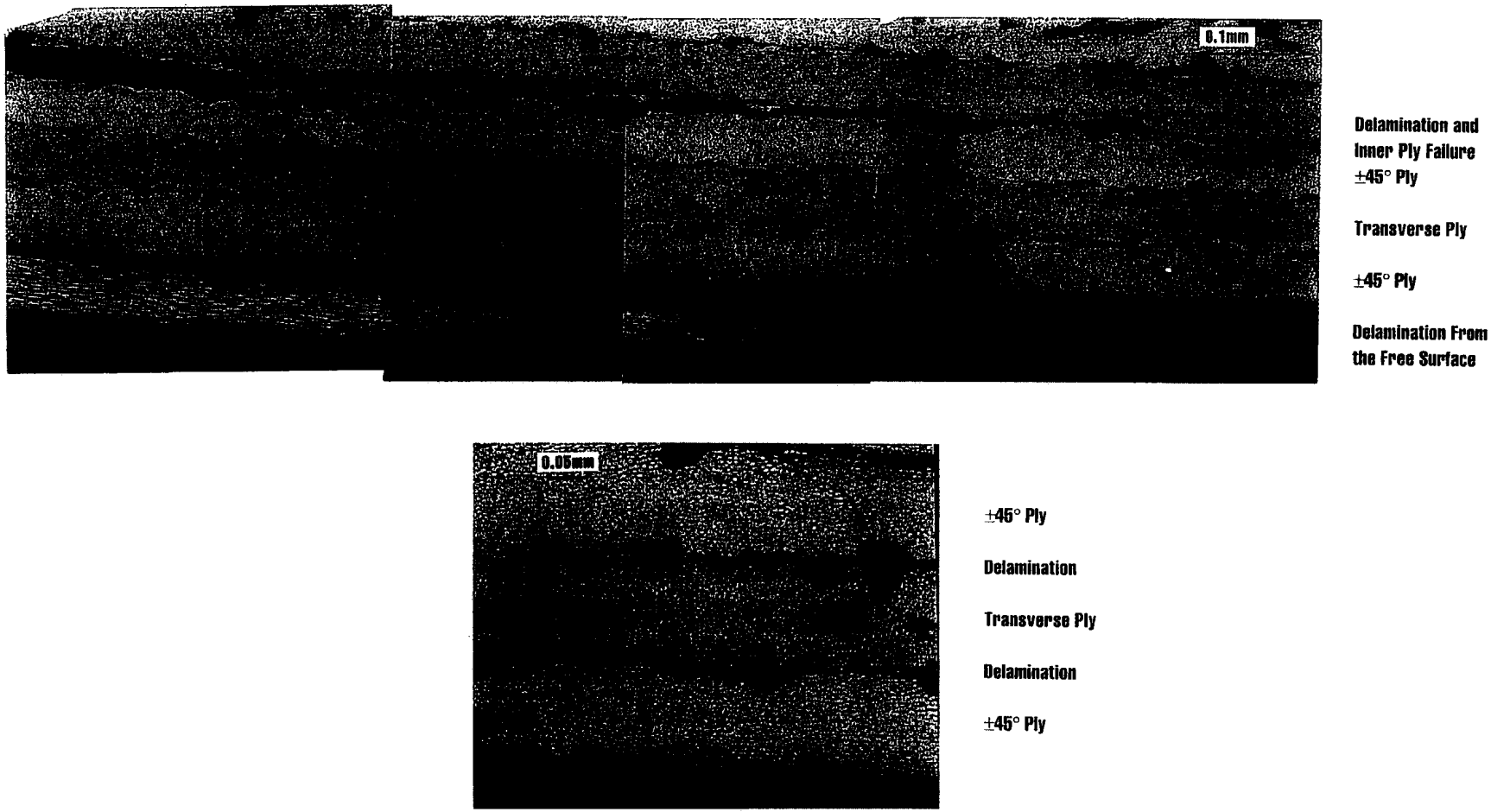

Fig. 1. Scanning electron micrograph of the tensile region of a four-point bend specimen illustrating delamination propagation along the fiber/matrix interfaces adjacent to the resin rich interlaminar region.

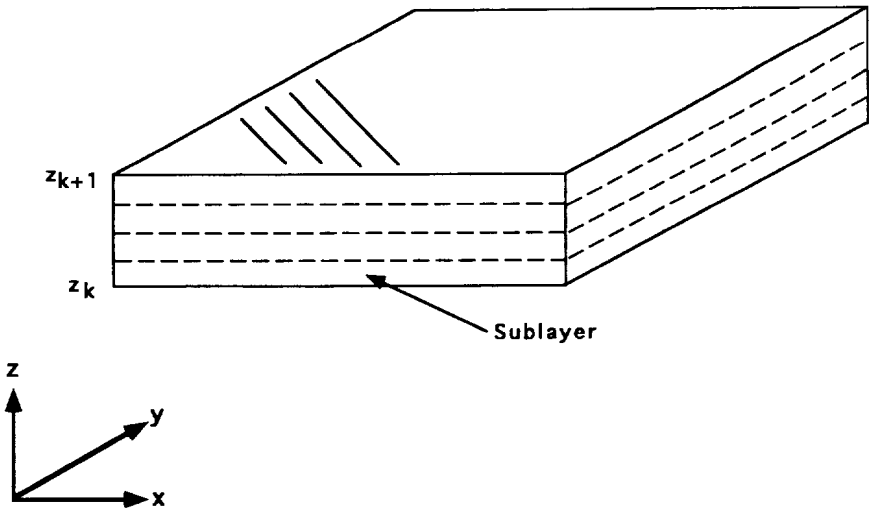

Fig. 2. Lamina geometry.

conditions. The second set of conditions represent the natural boundary conditions. In developing the above results the following definitions were used:

$$N_{rs}^{l} = \int_{z^k}^{z^{k+1}} \sigma_{rs}\phi^l \, dz \tag{7}$$

$$R_r^l = \int_{z^k}^{z^{k+1}} \sigma_{rz}\phi_{,z}^l \, dz \tag{8}$$

$$\tau_i^l = \sigma_{iz}\phi^l \Big|_{z^k}^{z^{k+1}} \tag{9}$$

$$T_i^l = \int_{z^k}^{z^{k+1}} t_i^* \phi^l \, dz \tag{10}$$

$$F_i^l = \int_{z^k}^{z^{k+1}} f_i \phi^l \, dz \tag{11}$$

and

$$I^{lm} = \int_{z^k}^{z^{k+1}} \rho\phi^l\phi^m \, dz \tag{12}$$

The terms τ_i^l are related to the interfacial tractions, the $N_{i\alpha}^l$ and R_i^l are force resultants. F_i^l correspond to body force effects, and I^{lm} are associated with inertia effects. The n_α are the unit normals along the boundary $\partial\Omega$.

It is emphasized that eqns (5) and (6) represent the equations of motion and appropriate boundary conditions, respectively, governing the response of a single computational layer. A computational layer may be thinner than a typical lamina in a structure, of equal thickness to a lamina, or it may be composed of several lamina. It is emphasized that these equations were developed independently of any structural considerations related to the laminated structure.

Explicit satisfaction of both the continuity of the interfacial tractions and the jump conditions on the interfacial displacements are utilized to couple the equations governing

the behavior of different layers to obtain the governing equations for the laminate. The jump in the displacement field across the interface between the k and $k+1$ layers is defined by

$$\Delta_i^k = u_i^{k+1} - u_i^k \tag{13}$$

where Δ_i^k is provided by an interfacial fracture constitutive law, eqn (1). The constitutive law for Δ_i^k can be heuristically or physically derived. The interfacial traction continuity condition is given by

$$\sigma_{iz}^k = \sigma_{iz}^{k+1} \tag{14}$$

To this point no restrictions have been placed on the approximation functions $\phi^j(z)$. The interfacial constraints, eqns (13) and (14), are most easily imposed if it is assumed that the $\phi^j(z)$ are Lagrange polynomials of order j. In particular, these polynomials have the following property:

$$\phi^j(z_i) = \delta_{ij} \tag{15}$$

where δ_{ij} is the Kronecker delta. This property makes the satisfaction of the interfacial constraints particularly simple. It is noted that the use of Lagrange polynomials is based on convenience only.

Substituting eqn (15) into eqn (9) gives

$$\begin{aligned}
(\tau_i^1)^k &= (\tau_i^1)^k = -\sigma_{iz}^k|_{z^k} \\
(\tau_i^N)^k &= (\tau_i^N)^k = \sigma_{iz}^k|_{z^{k+1}} \\
(\tau_i^j)^k &= 0 \quad \text{for} \quad j = 2, 3, \ldots, N-1.
\end{aligned} \tag{16}$$

Therefore, the interfacial traction continuity conditions, eqn (14), are satisfied by

$$(\tau_i^N)^k + (\tau_i^1)^{k+1} = 0. \tag{17}$$

The jump conditions for the displacements, eqn (13), at the kth interface are satisfied by

$$(V_i^1)^{k+1} - (V_i^N)^k = \Delta_i^k = f(\boldsymbol{\Delta}^k, \mathbf{t}^k) \tag{18}$$

for an Nth order Lagrange polynomial in z. Using eqn (16), the functional form for the displacement jump, eqn (1), can be rewritten as

$$\boldsymbol{\Delta}^k = f(\boldsymbol{\Delta}^k, \boldsymbol{\tau}^k). \tag{19}$$

Equation (19) is completely general but now the displacement jumps are expressed in a direct and consistent fashion as a function of the fundamental unknowns in the theory.

The equations of motion, eqn (5), subject to the interfacial constraints, eqns (17)–(19), in conjunction with the layer boundary conditions, eqn (6), form the system of equations governing the behavior of the laminated structure. The fundamental variables in the theory are the layer displacement coefficients (V_i^j) and the interfacial traction terms (τ_i^j). The above formulation has been carried out in a sufficiently general fashion that any constitutive law for the behavior of the layer or interface may be incorporated and, therefore, any evolution laws for the local effects can be consistently incorporated into the formulation.

Because the current theory is based on a discrete layer displacement formulation, it is useful to compare it to similar theories. The general theory of Reddy (1987) and the subsequent extensions developed by Lu and Liu (1992) have been shown to be effective

theories for modeling the behavior of perfectly bonded laminates. Under perfect bonding of the layers the current theory can be collapsed to either of these two theories (provided the appropriate polynomials are used). Both of the above theories have also been extended to modeling delamination behavior with good results (Liu *et al.*, 1994; Barbero and Reddy, 1991). The model used by Barbero and Reddy (1991) is based on a virtual crack extension methodology. Thus, this approach is used to analyze the influence of pre-existing cracks. The theory presented by Lui *et al.* (1994) is more general because it can model initiation and growth of delaminations. However, this theory is limited to the use of a linear delamination model. A linear delamination model is adequate for modeling the initial behavior of delaminated structures. It is not suitable for modeling delamination in the presence of large displacement jumps, which require a nonlinear interfacial constitutive model (cf. Needleman, 1990). The current formulation has several unique features that differentiate it from these theories in particular and other theories in general. The current formulation explicitly incorporates the interfacial tractions as fundamental variables. These interfacial tractions (τ_i^j) can be used in conjunction with any general interfacial constitutive law to determine the behavior of the structure in the presence of delaminations in an internally consistent fashion. Changing the interfacial fracture behavior does not require a reformulation of the current theory. This formulation is not restricted to modeling and preexisting delaminations and can predict the initiation and growth of delaminations anywhere within the structure. Finally, the growth of these delaminations is not in any way restricted to a given location or direction.

3. VALIDATION

The current theory is validated using exact solutions for the static cylindrical bending of laminated, cross-ply elastic plates. The perfectly bonded solution, i.e. no delaminations, is due to Pagano (1969). A modification of Pagano's solution to incorporate delaminations is used for the second part of the validation. The geometry of the cylindrical bending problem is given in Fig. 3.

The applied loading in Pagano's (1969) solution for the response of a cross-ply laminated structure is given by

$$\sigma_{zz}\left(x, z = \frac{h}{2}\right) = q_n \sin px \tag{20}$$

whereas $p = n\pi/L$. The solution is based on a stress function approach. The through the thickness variations of the stresses and displacements for orthotropic lamina is given by

$$f(z) = \sum_{i=1}^{4} A_i e^{m_i z}. \tag{21}$$

In the perfect bonding case, the coefficients A_i for the layers are determined by imposing continuity of tractions, eqn (14), and continuity of displacements, eqn (13), where $\Delta_j = 0$.

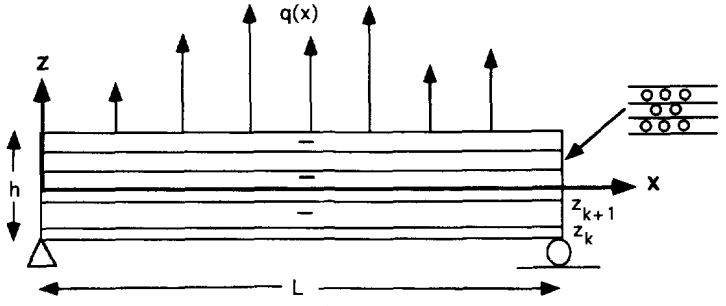

Fig. 3. Cylindrical bending geometry.

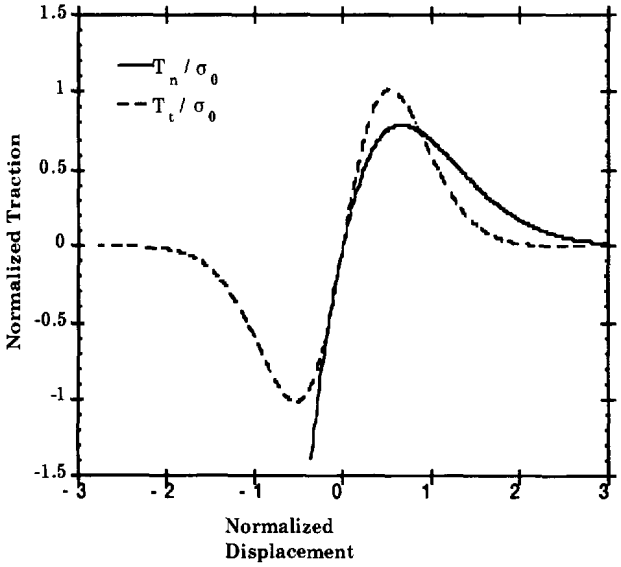

Fig. 4. General interfacial delamination constitutive law.

Orthogonality of the trigonometric functions imply that the functions $f(z)$ must be determined only for each harmonic used to express the transverse loading, eqn (20). A full eigenfunctions expansion is not necessary.

To carry out the analysis in the presence of delaminations the coefficients A_j for the layers are determined by imposing continuity of interfacial tractions, eqn (14), and the interfacial jump conditions, eqn (13). In this situation, $\Delta_i \neq 0$ in eqn (13). For the current study, the interfacial constitutive equations modeling the delamination behavior are given by

$$\Delta_z = R_n \sigma_{zz}$$
$$\Delta_x = R_s \sigma_{xz} \tag{22}$$

where R_n and R_s are constants representing the compliance for normal and shear delamination, respectively (Aboudi, 1991). In general, an interfacial fracture law will exhibit the behavioral characteristics displayed in Fig. 4. Both the interfacial normal (t_n) and shear (t_s) responses are provided in Fig. 4. Both responses exhibit an initial stiff behaviour up to a peak stress or strength. After this peak stress the behavior of the interface exhibits softening. Unlike the normal response, the shear response is symmetric. In general, the normal and shear responses are nonlinear and coupled (Needleman, 1987, 1990: Corigliano, 1993). The interfacial constitutive relations given by eqn (22), which are linear and uncoupled, represent the simplest possible model for delamination. This delamination model does not exhibit many of the characteristics consistent with delamination behavior, such as softening behavior. The model given by eqn (22) corresponds to the initial linear behavior observed in Fig. 4. The simplicity of the current delamination model maintains the orthogonality constraints on the function $f(z)$ mentioned earlier, i.e. it is only necessary to determine a function $f(z)$ for each loading harmonic. More complex nonlinear delamination models require the use of full eigenfunction expansions to model the delamination behavior regardless of the number of harmonics used in the representation of the loading at the top surface. It is noted that instead of the above model a more general nonlinear interfacial constitutive relation could be used. The influence of different delamination models will be addressed in a subsequent paper.

The nondimensionalizations proposed by Pagano are utilized in presenting the results.

$$\sigma_{xx}^* = \frac{\sigma_{xx}}{q_1} \quad \sigma_{zz}^* = \frac{\sigma_{zz}}{q_1} \quad \sigma_{xz}^* = \frac{\sigma_{xz}}{q_1} \quad u_x^* = \frac{E_T u_x}{h q_1} \quad u_z^* = \frac{100 E_T h^3 u_z}{L^4 q_1} \quad S = \frac{L}{H} \quad z^* = \frac{z}{h} \quad x^* = \frac{\sigma_{zz}}{L}.$$

The quantity S is the aspect ratio. The subscripts T and L denote quantities associated with the directions transverse to the fiber or along the fiber, respectively.

Table 1. Effective elastic properties of the Gr/Ep composite system

E_L (MPa)	172.4
E_T (MPa)	6.895
G_{LT} (MPa)	3.447
G_{TT} (MPa)	1.379
$\nu_{LT} = \nu_{TT}$	0.25

All of the validation cases are based on the following conditions. Each layer is a graphite fiber reinforced epoxy matrix (Gr/Ep) composite system with effective elastic material properties given in Table 1. The transverse loading at the top of the laminate is considered to be tensile and to consist of the first harmonic, $q_1 > 0$ and $q_n = 0$ for $n = 2, 3, \ldots, N$.

The first case considered is a unidirectional composite plate with the fibers aligned along the x axis, Fig. 3. In this case no delamination (perfect bonding) is assumed, i.e. $R_n = R_s = 0$. The results are presented for two versions of the current theory. The first version uses linear Lagrange polynomials and is termed the first order theory. In developing the first order theory results, the plate is subdivided into five computational regions or sublayers. The second version uses second order Lagrange polynomials and is called the second order theory. In the second order theory, four computational subdivisions are used to model the behavior of the laminate. The transverse deflection (u_3^*) evaluated at $x^* = 0.5$ vs the aspect ratio (S) as predicted by the two versions of the current theory and the exact solution is given in Fig. 5. As expected, the second order theory exhibits better agreement with the exact solution than the first order theory; however, both versions can be considered to correlate well with the exact results. The largest differences occur between the first order theory and the exact solution for small values of the aspect ratio. This difference between the first order theory predictions and the exact solution results is 2.5% for $S = 4.0$. Due to the fact that the approximate theory predicts stiffer transverse behavior as compared to the

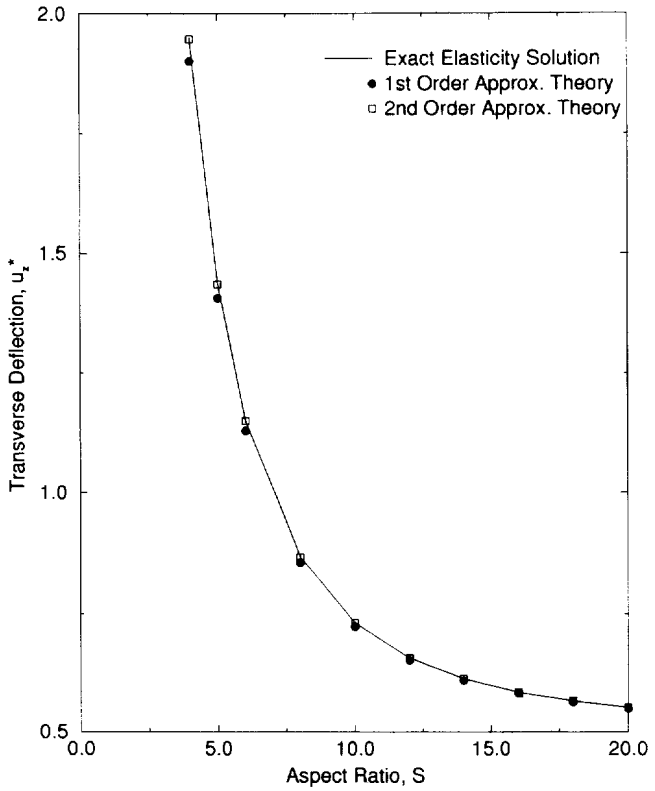

Fig. 5. Nondimensional transverse deflection (u_3^*) of an unidirectional plate as a function of the aspect ratio at $x^* = 0.5$.

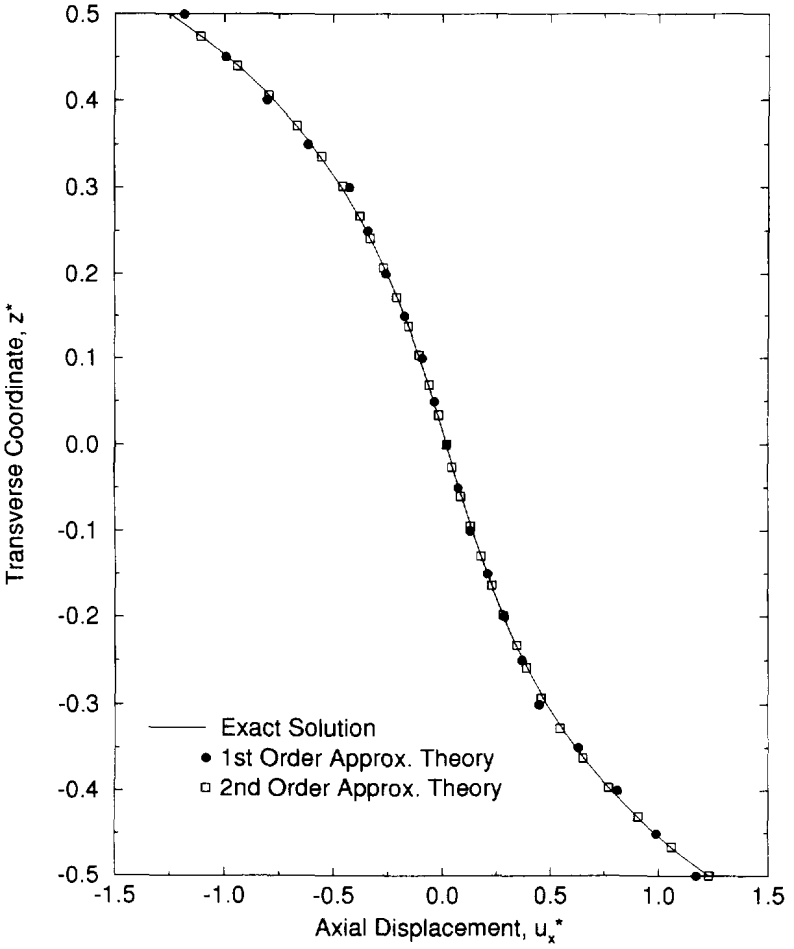

Fig. 6. Nondimensional axial diplacement (u_x^*) distribution through the thickness of a unidirectional plate at $x^* = 0$ for $S = 5.0$.

exact solution the approximate theory results fall below the exact solution. The comparisons for the inplane displacement (u_x^*) and stress (σ_{xx}^*) are provided in Figs 6 and 7, respectively. The correlations of both versions of the current theory with the exact results are excellent.

It is more important to consider the correlation between the transverse stresses (σ_{xz}^* and σ_{zz}^*) at the interfaces between layers (subdivisions) because it is these stresses that influence the delamination behavior. Values for the interfacial stresses obtained directly from the approximate solution and normalized by the exact solution stresses are presented in Table 2. Examination of the values shows that the approximate theory predictions differ from the exact solution results by less than 0.6% for both orders of the approximate theory. The correlation of the second order theory with the exact solution results is slightly better than the first order theory correlation. The excellent agreement between the interfacial

Table 2. Normalized interfacial transverse stresses for unidirectional plate

	$\sigma_{xz}^{App}\ \sigma_{xz}^{Ex}$		$\sigma_{zz}^{App}\ \sigma_{zz}^{Ex}$	
z^*	First order theory	Second order theory	First order theory	Second order theory
−0.30	1.0051		1.0051	
−0.25	—	0.9983		1.0015
−0.10	0.9996		1.0004	
0.00		0.9985		1.0000
0.10	1.0001		0.9994	
0.25	—	0.9984		0.9997
0.30	1.0059		0.9997	

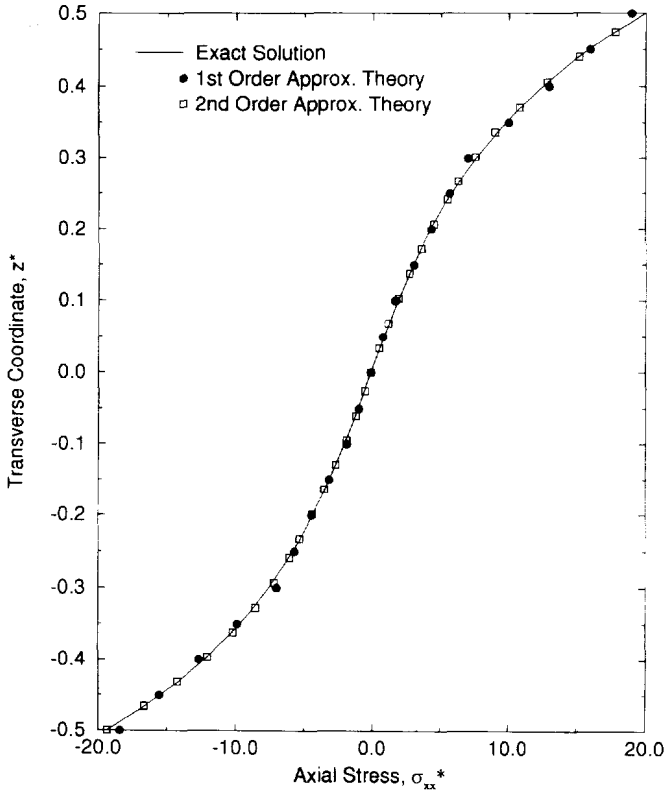

Fig. 7. Nondimensional axial stress (σ^*_{xx}) distribution through the thickness of a unidirectional plate at $x^* = 0.5$ for $S = 5.0$.

stresses predicted by both forms of the approximate theory and the exact solution is important for the subsequent application of the theory to model the response of the plate in the presence of delaminations.

Application of the current theory to the analysis of perfectly bonded plates with more lamina results in predictions (not provided) which exhibit the same level of agreement with the exact solution as seen above. It is noted that as the number of lamina increases the structural behavior can generally be accurately modeled by treating each lamina as a single layer. Thus, it may not be necessary to subdivide each lamina into several layers in these situations.

The second case considered, which incorporates delaminations, is that of a two layer cross-ply laminate. The bottom layer has the fibers aligned perpendicular to the x-z plane while the fibers are aligned along the x-axis in the top layer. All approximate results are based on the use of the second order theory. Results are presented for two different behaviors. The first set of results correspond to perfect bonding ($R_n = R_s = 0.0$). The second set of results model delamination using $R_n = 3.3811 \times 10^{-06}$ m/MPa and $R_s = 2.2310 \times 10^{-06}$ m/MPa. These values for the R_n and R_s were determined from rough micromechanics calculations based on experimental data for the initial delamination response of a unidirectional material. The results obtained from the approximate theory are based on two different discretization schemes within a lamina. In the first discretization scheme, each lamina is modeled as a single layer. This scheme is used to generate results for both the perfect bonding and the delamination response. The second discretization scheme models each lamina using two subdivision. In the second discretization scheme, the layer or subdivision furthest from the interface (in both lamina) has a nondimensional thickness of 0.4 (with respect to the laminate thickness). The sublayer closest to the interface (in both lamina) has a nondimensional thickness of 0.1. The second discretization scheme is used to model the behaviour of the laminate in the presence of delaminations only.

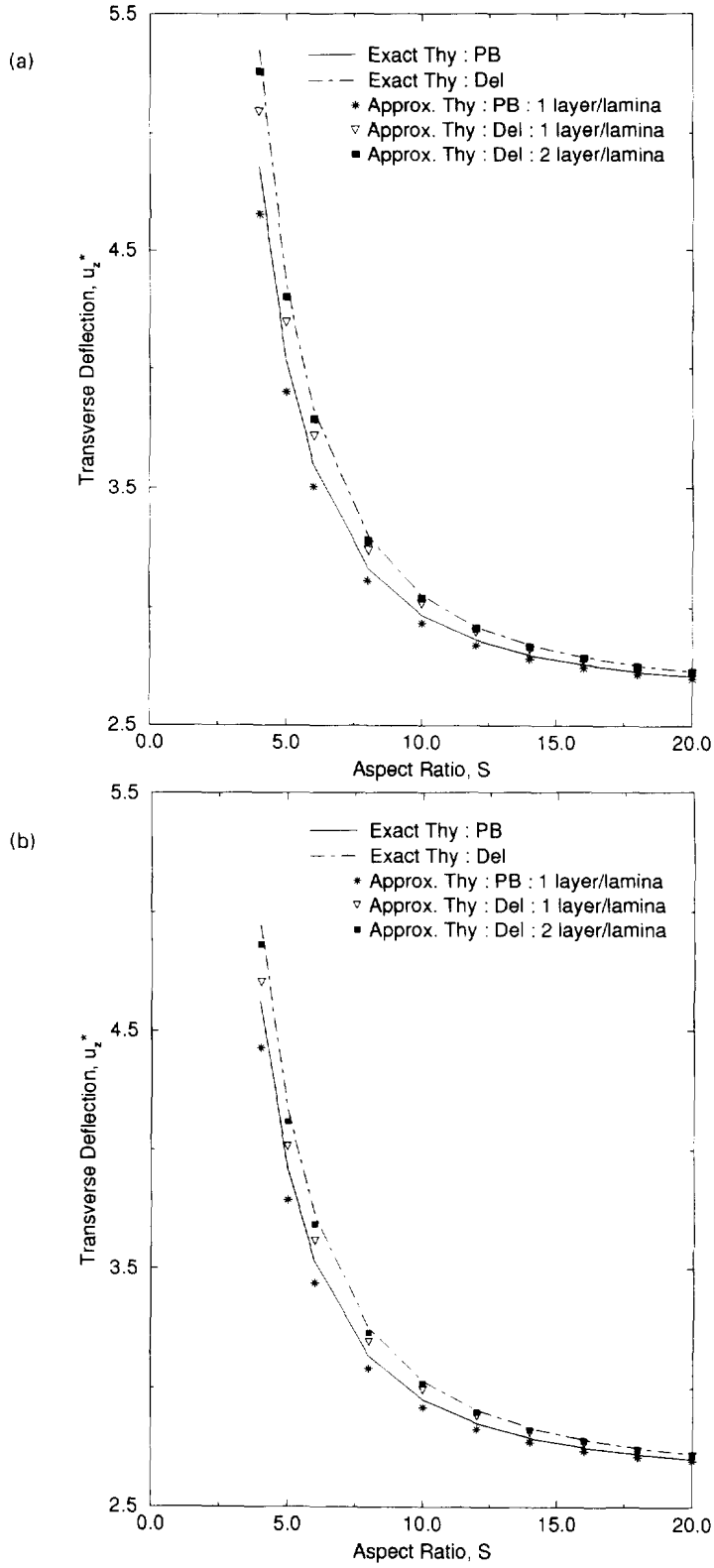

Fig. 8. Nondimensional transverse deflections (u_z^*) of a two layer cross ply plate as a function of the aspect ratio at $x^* = 0.5$: (a) top surface; (b) bottom surface.

The variation of the transverse deflection (u_z^*) at the top surface of the laminate as a function of the aspect ratio (S) is given in Fig. 8(a) for both the perfect bonding (PB) and delamination (Del) cases. The corresponding response at the bottom surface is given in Fig. 8(b). The deflections in both figures are evaluated at the midsection ($x^* = 0.5$).

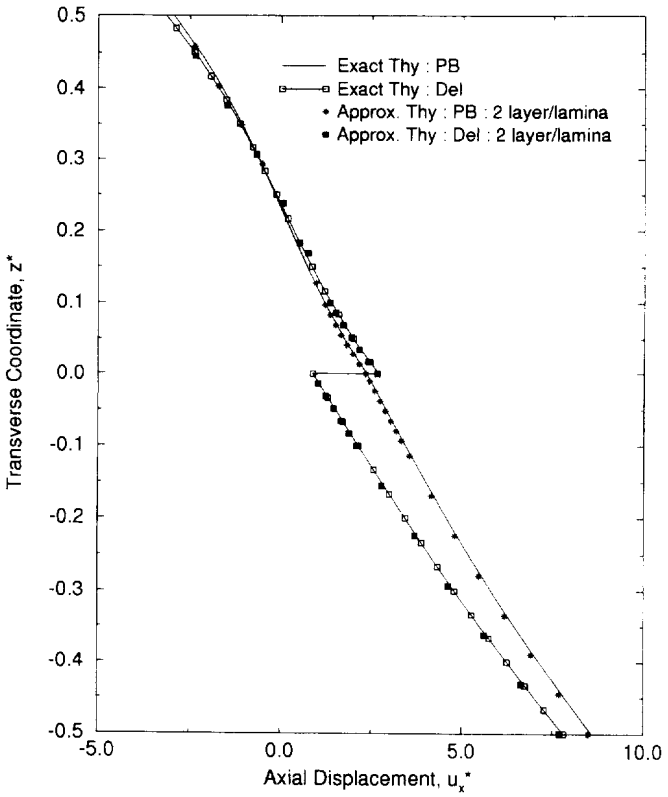

Fig. 9. Nondimensional axial displacement (u_x^*) distribution through the thickness of a two layer cross ply plate at $x^* = 0$ for $S = 5.0$.

Examination of both Fig. 8(a) and (b) indicates that the approximate theory accurately captures the trends of the behavior for both the perfectly bonded and delaminated plates. In particular, both the exact solution and approximate theory (for both discretization schemes) predict similar differences between the magnitudes of the transverse deflections at the top and bottom surfaces of the plate for both the perfectly bonded and delaminated situations. The differences in the magnitudes of the deflections at the top and bottom surfaces are largest for the delamination case and are 4.7% for $S = 4$ for both the exact and approximate theories. As S increases both theories predict that the relative differences between the deflections at the top and bottom surfaces decrease. For larger values of S the approximate and exact predictions converge and both theories predict negligible relative differences between the top and bottom surface deflections. Use of the second discretization scheme (where each lamina is modeled using two sublayers) gives better correlation with the exact theory for the actual values of the displacements at both the top and bottom surfaces of the plate. The largest difference is about 2% and occurs at $S = 4$. Finally, both the exact and approximate theories predict that the influence of delamination (as compared to the perfectly bonded response) is the largest for small values of S. The exact solution and the approximate solution based on two sublayers per lamina predict differences between the perfect bonded and delaminated cases of about 10% for the displacement at the top surface for an S of 4. Because of the above results, the following comparisons for the two lamina cross ply laminate are based only on the second discretization scheme.

The distribution of the inplane displacement (u_x^*) as a function of the transverse coordinate z^* at the edge of the plate ($x^* = 0.0$) is presented in Fig. 9. This position corresponds to the location of the largest axial displacements. It may be seen that the overall correlation between the approximate theory and the elasticity solution is excellent in both the perfect bonding and delaminations cases. In particular, the magnitudes and gradients in the displacement distribution are predicted accurately by the approximate theory. The correlation between the exact and approximate predictions for the displacement jump at the interface is excellent with no discernible difference in the magnitudes of the

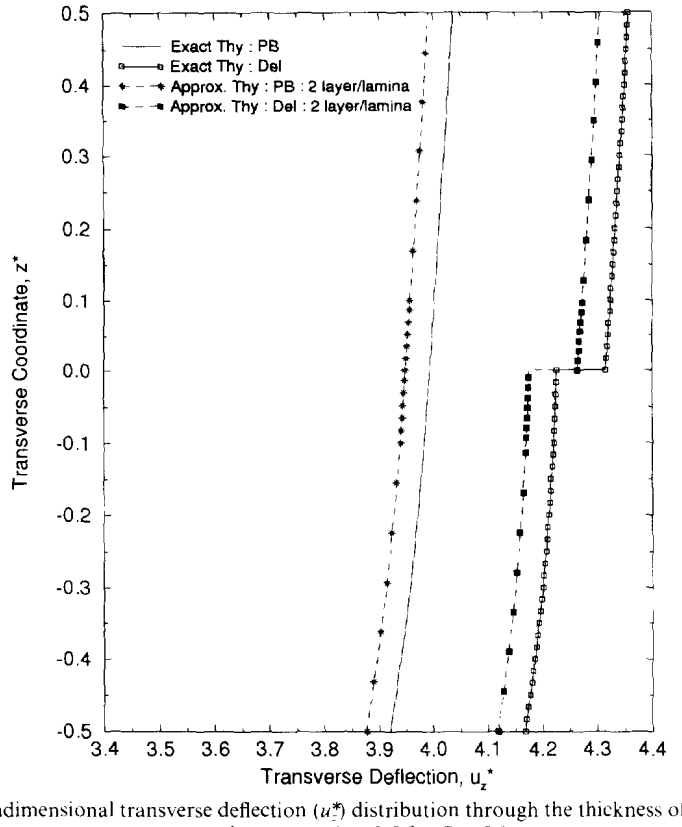

Fig. 10. Nondimensional transverse deflection (u_z^*) distribution through the thickness of a two layer cross ply pate at $x^* = 0.5$ for $S = 5.0$.

jumps. The magnitude of the displacement jump is 16.7% of the difference between the axial displacements at the top and bottom of the plate. The influence of the delamination is to shift the axial displacement in the lamina away from the values predicted in the perfect bonding case. These shifts are greatest at the location of the delamination, i.e. at the interface between the lamina. Comparison of the shifts in the displacement between the laminate indicates that the largest shift occurs in the bottom lamina. This shift is about 3.8 times the corresponding shift in the top layer. In the top lamina the deviations from the perfect bonding case decrease as the distance from the interface increases. In the bottom lamina significant shifts in the axial displacement from the perfect bonding result occur throughout the thickness of the lamina, although the magnitude of the shift decreases as the distance from the interface increases.

The distribution of the transverse displacement (u_z^*) as a function of z^* at the midsection ($x^* = 0.5$) is presented in Fig. 10. The overall correlation of the approximate solution predictions with the results of the exact solution is excellent for both the perfect bonding and delamination cases. The approximate solution predicts a stiffer response in the form of decreased transverse deflections as compared to the exact solution. This increased stiffness of the approximate solution results in shifts in the distributions of the deflection of less than 1.0% for both the perfect bonding and delamination cases. The gradients of the transverse displacement predicted by both theories are in excellent agreement. The displacement jumps predicted by the elasticity solution and the approximate plate solution are the same. The displacement jump due to the delamination (in both theories) represents 77% of the total change in the transverse deflection from the top surface to the bottom surface.

The distribution of the inplane stress (σ_{xx}^*) as a function of z^* at the midplane ($x^* = 0.5$) is presented in Fig. 11. The correlation between the approximate solution and the exact solution results are excellent with the differences being negligible at all points. The presence of the delamination primarily influences the response (with respect to the perfect bonding case) of the top layer with only minor changes in the response of the bottom layer. The

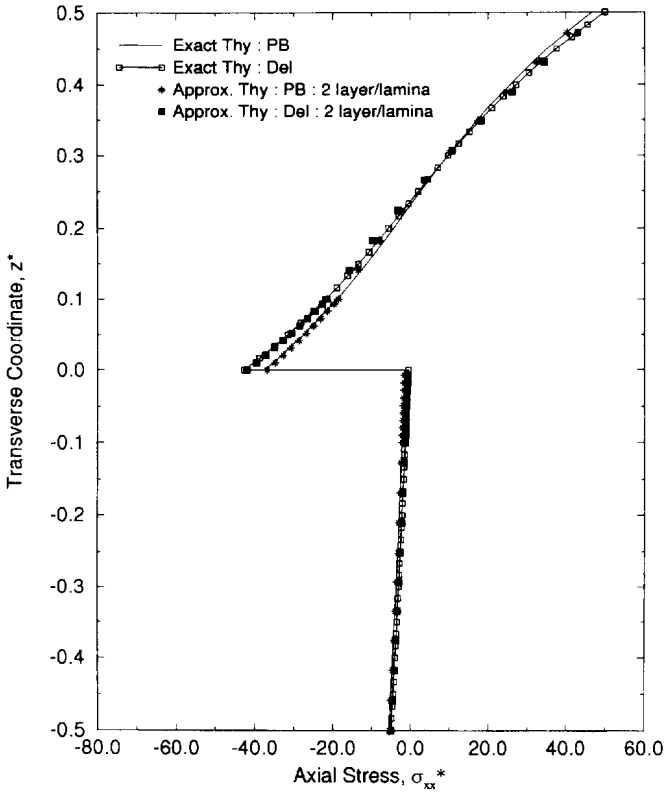

Fig. 11. Nondimensional axial stress (σ^*_{xx}) distribution through the thickness of a two layer cross ply plate at $x^* = 0.5$ for $S = 5.0$.

delamination causes the stress at the top surface of the laminate to increase in magnitude by 6.0% for the delamination case as compared to the perfect bonding case. At the interface between the lamina ($z^* = 0$) the delamination causes the stress to become more negative than the perfect bonding case with a corresponding change in magnitude of 10% for the delamination case. Examination of the interfacial stresses as obtained directly from the approximate theory shows that the deviations from the exact solution results are less than 0.5%.

Finally, the response of a 5 layer laminate consisting of plies of unequal thickness is considered. The stacking sequence is given by [0°/90°/0°/90°/0°] where 0° is along the x axis. The corresponding nondimensional thicknesses (starting with the bottom layer) are given by 0.1, 0.2, 0.2, 0.25 and 0.25, respectively. The delamination compliances are given by $R_n = 3.3811 \times 10^{-07}$ m/MPa and $R_s = 2.2310 \times 10^{-07}$ m/MPa for all interfaces. The perfect bonding responses as predicted by both theories also are included for reference. In the approximate analysis, each lamina is modeled as a single layer using the second order theory. Consideration of the axial displacement (u^*_x) at $x^* = 0.0$, Fig. 12, indicates that the approximate theory gives excellent agreement with the exact solution throughout the thickness of the laminate with differences of less than 0.5% at all points. The displacement jumps at the interfaces are modeled accurately by the approximate approach. The presence of the delamination has a dramatic influence on the distribution of the axial displacement. In particular, the gradients in the displacement distributions are significantly shifted from the perfect bonding case. As seen previously, the presence of the delaminations causes the axial displacement field in a layer to shift away from the response under perfect bonding. These shifts are not evenly distributed between the layers adjacent to a delamination. Furthermore, these shifts occur in opposite directions for a given layer in the presence of delaminations at the top and bottom of the layer. Very good correlation between the approximate and exact approaches is obtained for the transverse displacement (u^*_z) distribution through the thickness at the midplane, Fig. 13. As in the previous results, the

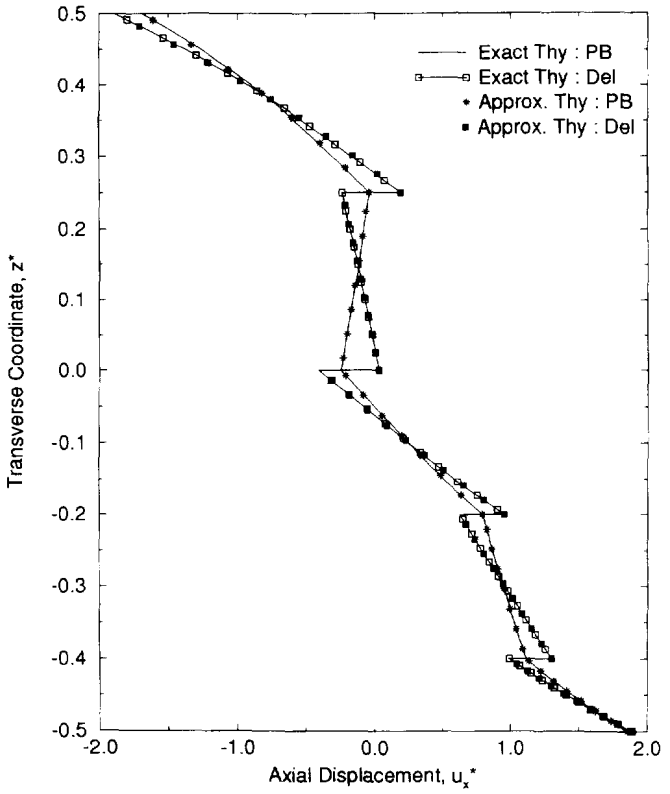

Fig. 12. Nondimensional axial displacement (u_x^*) distribution through the thickness of a five layer cross ply plate at $x^* = 0$ for $S = 5.0$.

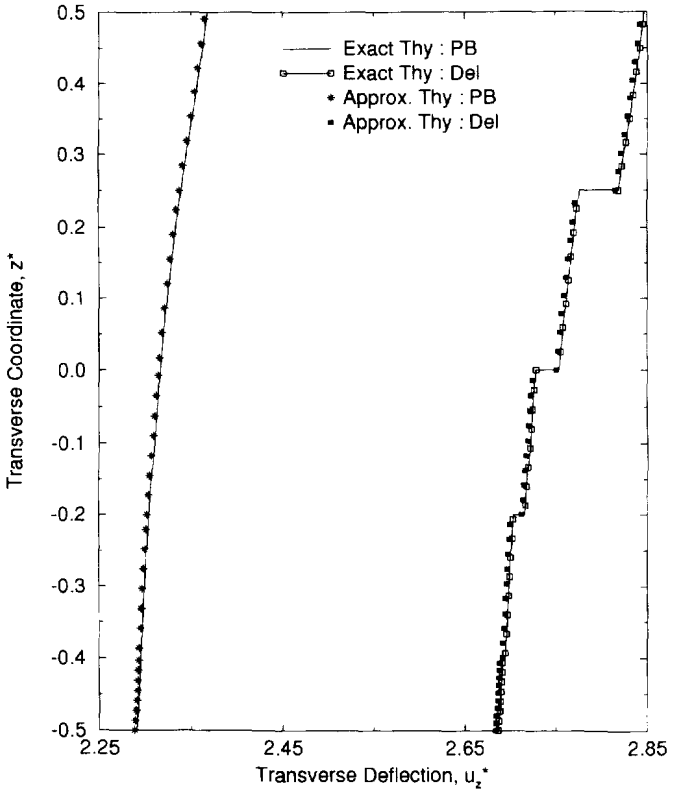

Fig. 13. Nondimensional transverse deflection (u_z^*) distribution though the thickness of a five layer cross ply plate at $x^* = 0.5$ for $S = 5.0$.

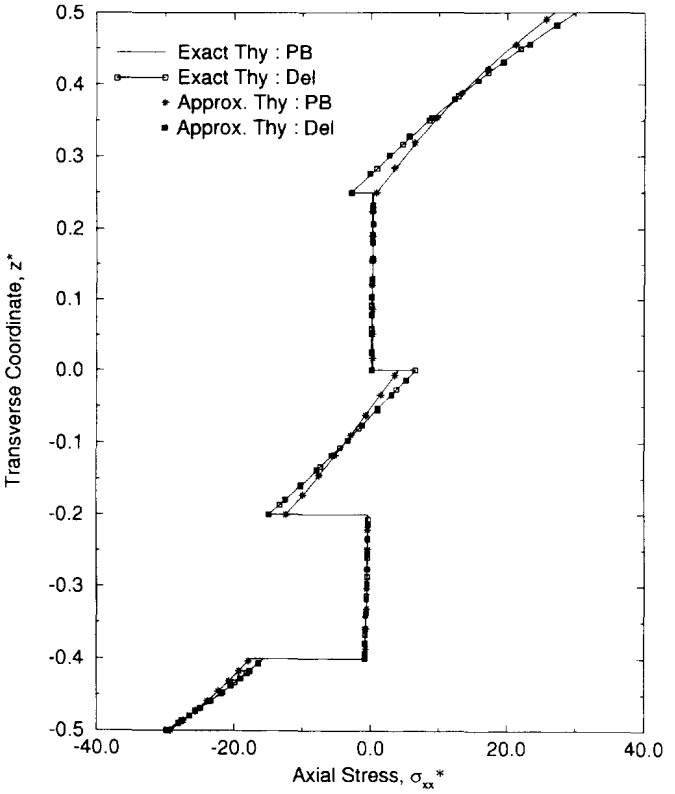

Fig. 14. Nondimensional axial stress (σ^*_{xx}) distribution through the thickness of a five layer cross ply plate at $x^* = 0.5$ for $S = 5.0$.

approximate theory is stiffer than the exact theory as exemplified by the small shift to lower values in the deflection. However, this shift is less than 0.5% throughout the thickness of the laminate. The presence of the delamination causes the transverse deflection distribution to shift 15% to a higher magnitude as compared to the perfect bonding response. Finally, excellent agreement between the two theories is obtained for the axial stress distribution (σ^*_{xx}) at the midplane, Fig. 14. As seen previously, the presence of the delaminations has the strongest influence on the stress state in the 0 plies. The trends in the behavior of the axial stress due to the influence of the delaminations are the same as observed in the axial displacement component. The transverse interfacial stresses as evaluated directly from the approximate theory exhibit differences from the exact solution of less than 0.5%. Comparison of the displacement jumps in these approximate results with those of the previous results shows that the addition of more lamina results in larger jumps in the interfacial displacements when both the applied loading and the interfacial parameters, R_u and R_v are held constant.

4. SUMMARY AND CONCLUSIONS

A theory of laminated plates based on a discrete layer analysis has been presented. The model implements a generalized displacement formulation at the lamina level. The governing equations for the lamina are derived using variational principles. The equations governing the behavior of the laminate are developed by coupling the layer equations through both the interfacial traction continuity conditions and the interfacial displacement jump conditions. The fundamental variables in the governing equations are the displacements in each layer (V_i^j) and the interfacial traction terms (τ_i^j). Finally, the current theory is not limited to the analysis of thick or thin laminated plates.

The jumps in the interfacial displacements are expressed in an internally consistent fashion as functions of τ_i^j, i.e. without recourse to integration of the equilibrium equations.

In particular, integration of the pointwise equilibrium equations is not feasible in the presence of delaminations due to the interdependence of the deformations and the delaminations. Additionally, under dynamic loading conditions use of the pointwise equilibrium equations is also not appropriate and the interfacial tractions must be obtained directly. Therefore, the use of the τ_i^I as fundamental unknowns is important in predicting delamination behavior in an internally consistent fashion. The formulation has been carried out in a sufficiently general fashion that any constitutive model for the delamination behavior at the interfaces may be incorporated into the theory. No restrictions on the size, location, distributions, or direction of growth of the delaminations have been imposed in the theory. As a result, the current theory can predict the initiation and growth of delaminations at any interface as well as any interactive effects between delaminations at different locations within the structure.

The current formulation can incorporate any constitutive relations for the response of the material in the individual layers. This ability coupled with the general layer-wise formulation allows the local effects such as inelastic deformations and damage within layers to be modeled to any desired accuracy.

It is noted that the computational layers on which the analysis is based can consist of subregions of a lamina, an entire lamina, or several lamina. The ability to incorporate several lamina into a layer would be beneficial in situations where homogenization of a repeated stacking sequence can be used. While the current theory has been formulated for laminated composite structures it can be used to model the behavior of monolithic plates in the presence of local damage, such as cracking, due to its ability to analyze subregions of the plate accurately.

The current theory has been validated by comparing the predictions to the results obtained from an exact solution. The exact solution is a modification of Pagano's elasticity solution for cylindrical bending to include the effects of delaminations. The agreement between the approximate and exact theories for all cases considered was excellent. The approximate theory very accurately captured the trends caused by the delaminations observed in the exact solution predictions. In most cases, the differences between the approximate theory and the exact solution were less than 1.0%. Furthermore, by incorporating the interfacial traction effects as a fundamental unknown in the formulation very accurate predictions for these effects were obtained in a consistent fashion directly from the theory (i.e. without recourse to use of integration of the pointwise equilibrium equations). This in turn resulted in very accurate (internally consistent) predictions for the jumps in the interfacial displacements. The displacement jumps due to delamination can represent a significant portion of the change in the displacements through the thickness of the laminate. Therefore, these jumps can have a strong effect on both the microscopic and macroscopic response of the laminate. It was observed that the axial displacement and stress at a delaminated interface shift away from the corresponding values predicted under perfect bonding. When delaminations were present at both the top and bottom of a layer, the shifts at these locations were in opposite directions. The influence of the delaminations on the axial stress was largest in plies where the fibers were orientated along the x axis due to the high ply stiffness in this direction. The effects on the stress distributions in plies with the fibers aligned along the y axis (90) were negligible. The transverse deflections within the plate increased in the presence of delaminations. Additionally, the difference between the deflections at the top and bottom surface increased in the presence of delamination. For the cases considered, increasing the number of layers, and, therefore, the number of delaminations resulted in larger jumps in the interfacial displacements for a given set of delamination parameters R_n and R_s and loading magnitude. Finally, it was observed that the influence of the delaminations was greatest for small values of the aspect ratio, S. This influence significantly decreased as S increased.

A number of extensions of this research are currently being pursued. They include the implementation of physically based interface models. A finite-element formulation for the dynamic response of laminated plates also is under investigation. Comparison of the theory with experimental data for double cantilevered beams also will be considered. These issues will be addressed in a forthcoming paper.

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APPENDIX

For the convenience of the reader, the governing equations for both the exact solution and the approximate laminated plate solution are presented in this appendix. The exact solution equations are written in consistent notation with the rest of the paper. In both cases the solutions have been specialized to cross-ply laminates. The plate is an infinite strip which is simply supported at both ends, Fig. 3.

Exact solution

The exact solution for the static behavior of a delaminated plate is based on Pagano's solution for perfectly bonded plates (1969). The solution for a lamina is developed under the assumption that a state of plane strain exists. This assumption implies that a state of generalized plane stress exists. Additionally, the boundary conditions and the fact that the plate is an infinite strip imply that all quantities depend only on the x-z coordinates. Substitution of these conditions into the equlibrium equations results in two governing differential equations for the stresses. A solution to the governing system of equations is obtained through the use of Airy's stress function defined by

$$\sigma_{xx}^{(k)} = \sum_{n=1}^{\infty} f_{kn}''(z) \sin p_n x$$

$$\sigma_{zz}^{(k)} = \sum_{n=1}^{\infty} p_n^2 f_{kn}(z) \sin p_n x$$

$$\sigma_{xz}^{(k)} = -\sum_{n=1}^{r} p_n f_{kn}'(z) \cos p_n x \tag{A.1}$$

where the prime denotes differentiation with respect to z and $p_n = (n\pi)/L$. Substitution of the above equations into the governing differential equations gives the following form for the stress function $f_i(z)$:

$$f_{kn}'(z) = \sum_{j=1}^{4} A_{jkn} \exp(m_{jkn} z_k) \tag{A.2}$$

where the m_{jkn} are functions of the material properties and the A_{jkn} are the constants to be determined by the interfacial conditions. The appropriate traction continuity conditions are

$$\sigma_{iz}^{(k)}\left(x, \frac{h_k}{2}\right) = \sigma_{iz}^{(k+1)}\left(x, -\frac{h_{k+1}}{2}\right). \tag{A.3}$$

The displacement jump conditions are given by eqn (13) which is repeated here for convenience

$$\Delta_i^k = u_i^{k+1} - u_i^k \tag{A.4}$$

where the constitutive relation for Δ_i^k is given by eqn (22) in the current study.

Approximate plate solution

For the cylindrical bending problem of the approximate theory, similar assumptions as those used in the exact solution are employed. However, the solution is developed in terms of the displacement components rather than a stress function. The displacement field which satisfies the boundary conditions is given by

$$\begin{aligned} u_x &= \sum_{n=1}^{r} V_x^{jn} \phi^j(z) \cos p_n x \\ u_y &= 0 \\ u_z &= \sum_{n=1}^{r} V_z^{jn} \phi^j(z) \sin p_n x. \end{aligned} \tag{A.5}$$

Substituting the above displacement field into the governing differential equations for the plate theory, eqns (5), (7)–(10), (17), and (18), gives a system of equations for the layer in the form

$$\boldsymbol{\tau} - \mathbf{K}\mathbf{V} = 0 \tag{A.6}$$

where

$$\begin{aligned} \boldsymbol{\tau}^n &= (\tau_x^{1n}, 0, \tau_x^{3n}, \tau_z^{1n}, 0, \tau_z^{3n})^{\mathrm{T}} \\ \mathbf{V}^n &= (V_x^{1n}, V_z^{1n}, V_x^{2n}, V_z^{2n}, V_x^{3n}, V_z^{3n})^{\mathrm{T}} \\ \mathbf{K}^n &= \begin{bmatrix} a_{11} & b_{11} & a_{12} & b_{12} & a_{13} & b_{13} \\ a_{21} & b_{21} & a_{22} & b_{22} & a_{23} & b_{23} \\ a_{31} & b_{31} & a_{32} & b_{32} & a_{33} & b_{33} \\ \hat{a}_{11} & \hat{b}_{11} & \hat{a}_{12} & \hat{b}_{12} & \hat{a}_{13} & \hat{b}_{13} \\ \hat{a}_{21} & \hat{b}_{21} & \hat{a}_{22} & \hat{b}_{22} & \hat{a}_{23} & \hat{b}_{23} \\ \hat{a}_{31} & \hat{b}_{31} & \hat{a}_{32} & \hat{b}_{32} & \hat{a}_{33} & \hat{b}_{33} \end{bmatrix}. \end{aligned} \tag{A.7}$$

The individual terms in the matrix $\mathbf{K}^n$ are functions of the polynomials $\phi^j(z)$, the material properties, the geometric properties, and the harmonic number of the associated sine function.

$$\begin{aligned} a_{ijm} &= p_n^2 C_{11} \int_{z_k}^{z_{k+1}} \phi_i \phi_m \, \mathrm{d}z + C_{55} \int_{z_k}^{z_{k+1}} \phi_i' \phi_m' \, \mathrm{d}z \\ b_{ijm} &= -\left(C_{13} \int_{z_k}^{z_{k+1}} \phi_i \phi_m' \, \mathrm{d}z - C_{55} \int_{z_k}^{z_{k+1}} \phi_m \phi_i' \, \mathrm{d}z \right) p_n \\ \hat{a}_{ijm} &= p_n^2 C_{55} \int_{z_k}^{z_{k+1}} \phi_i \phi_m \, \mathrm{d}z + C_{33} \int_{z_k}^{z_{k+1}} \phi_i' \phi_m' \, \mathrm{d}z \\ \hat{b}_{ijm} &= -\left(C_{13} \int_{z_k}^{z_{k+1}} \phi_i \phi_m' \mathrm{d}z - C_{55} \int_{z_k}^{z_{k+1}} \phi_m \phi_i' \, \mathrm{d}z \right). \end{aligned} \tag{A.8}$$

This system in conjunction with the interfacial conditions, eqns (17)–(19) and (22), provides the necessary equations for the determination of the displacement terms V_i^{jn}.